AN EFFICIENT SIXTH-ORDER NUMERICAL BLOCK APPROACH FOR DIRECT INTEGRATION OF SECOND ORDER NON-LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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O. O. Olanegan*, B. G. Ogunware and E. O. Omole
Department and Mathematics, Federal University Oye Ekiti, Ekiti State, Nigeria
*Corresponding author: ola3yemi@gmail.com

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Abstract: An efficient hybrid block method with variable step size for solving second-order nonlinear initial value problems (IVPs) is proposed in this paper. The scheme was developed through collocation and interpolation of power series approximate solution to give a continuous linear multistep method. The evaluation of the continuous method at the grid and off grid points produced the discrete block method. To validate the proposed method, the fundamental properties of the block method are examined. The accuracy of the developed method is confirmed by assessing it with the existing methods to solve three non-linear second order initial value problems and the outcome proved that the 1-point hybrid block method is efficient and appropriate for integrating second-order initial value problem.

Keywords: Collocation & interpolation, efficiency, hybrid block method, initial value problems

Introduction

Ordinary Differential Equations (ODEs) has capacity to express many complex occurrences in biological science, physical sciences, fluid mechanics, optics, and engineering to models. Differential equations of second order can be seen in broad areas of applications of science and engineering (Santra *et al.* 2020). Some of these differential equations are usually expressed in models. Efforts have been made to develop methods to solve the resulting models.

This article focuses on a new numerical approach for solving directly nonlinear second-order initial value problems (IVPs) with the presence of y' in the form;

$$y'' = f\left(x, y, y'\right),\ \ y\left(a\right) = y_0,\ \ y'\left(a\right) = y',\ \ x \in \left[a, b\right] \quad (1)$$

Several existing numerical procedures are available in literatures for the direct solution of equation (1) directly and these techniques depend on many factors such as speed of convergence, computational expenses, data storage requirement and accuracy (Akeremale *et al.* 2020; Ogunware and Omole, 2020; Ukpebor *et al.*, 2019; Olanegan, 2018; Waeleh and Majid, 2017; Adoghe and Omole, 2017; Omar and Kuboye, 2016).

Authors have also used the method of collocation and interpolation of the power series approximation to generate continuous linear multistep method to formulate Block methods for approximating the numerical solution of (1) with the benefit of improving the level of accuracy exhibited in comparison to earlier existing methods as discussed by Mohammed and Adeniyi (2015), Badmus (2014), Odekunle *et al.* (2014) and many other scholars.

Hasan and Zhu (2009) developed a modified Adomian Decomposition Method for solving second-order ordinary differential equations with constant coefficients. The method was used as a good approximation to linear and nonlinear problems.

Nevertheless, this article Utilize the benefits associated with the block method to formulate a new hybrid block method by using its continuous formulation nature to produce a number of discrete method. After evaluation to the discrete scheme, the block is equally obtained from the derivative of the continuous method to form our block method that has better performance than some existing methods.

Formulation of the Method

Let (2) be power series approximate solution to (1) of the form:

$$y\left(x\right) = \sum_{j=0}^{\kappa+\nu-1} a_j x^j \quad (2)$$

Where $\kappa = 2$ and $\nu = 6$ as the number of interpolation and collocation points respectively, which is seen as an estimated solution to (1)

The first and second derivatives of (2) are given as

$$y' = \sum_{j=1}^{\kappa+\nu-1} j a_j x^{j-2} \quad (3)$$

$$y'' = \sum_{j=2}^{\kappa+\nu-1} j(j-1) a_j x^{j-2} \quad (4)$$

Substituting (4) into (1) produces a differential system of the form

$$y'' = \sum_{j=0}^{\kappa+\nu-1} j\left(j-1\right) a_j x^{j-2} = f\left(x, y, y'\right) \quad (5)$$

Collocating (5) at $x = x_{n+j},\ j = 0\left(\frac{1}{6}\right)1$ and interpolating (2) at $x = x_{n+j},\ j = \frac{1}{3}, \frac{1}{2}$ produce a system of linear equations with unknown $a's$ which are solved by Gaussian elimination method and then substituted into (2) to produce an implicit continuous method as follows:

$$y\left(t\right) = \alpha_{\frac{1}{3}}\left(t\right) y_{n+\frac{1}{3}} + \alpha_{\frac{1}{2}}\left(t\right) y_{n+\frac{1}{2}} + h^2\left[\sum_{\eta=0}^{1} \beta_\eta\left(t\right) f_{n+\eta} + \beta_{\frac{1}{6}}\left(t\right) f_{n+\frac{1}{6}} + \beta_{\frac{1}{3}}\left(t\right) f_{n+\frac{1}{3}} + \beta_{\frac{1}{2}}\left(t\right) f_{n+\frac{1}{2}} + \beta_{\frac{2}{3}}\left(t\right) f_{n+\frac{2}{3}} + \beta_{\frac{5}{6}}\left(t\right) f_{n+\frac{5}{6}}\right] \quad (6)$$

Performing a transformation on (6) using $t = \frac{x - x_n}{h}$, $\frac{dt}{dx} = \frac{1}{h}$. The coefficients of y_{n+j} and f_{n+j} are obtained with respect to t as:

$$\alpha_{\frac{1}{3}}(t) = (-6t + 3)$$

$$\alpha_{\frac{1}{2}}(t) = (6t - 2)$$

$$\beta_0(t) = \left[\frac{81}{70}t^8 - \frac{27}{5}t^7 + \frac{21}{2}t^6 - \frac{441}{40}t^5 + \frac{203}{30}t^4 - \frac{49}{20}t^3 + \frac{1}{2}t^2 - \frac{36737}{725760}t + \frac{1291}{725760}\right]$$

$$\beta_{\frac{1}{6}}(t) = \left[-\frac{243}{35}t^8 + \frac{216}{7}t^7 - \frac{279}{5}t^6 + \frac{261}{5}t^5 - \frac{261}{10}t^4 + 6t^3 - \frac{9859}{40320}t + \frac{1217}{40320}\right]$$

$$\beta_{\frac{1}{3}}(t) = \left[\frac{243}{14}t^8 - \frac{513}{7}t^7 + \frac{1233}{10}t^6 - \frac{4149}{40}t^5 + \frac{351}{8}t^4 - \frac{15}{2}t^3 - \frac{14213}{241920}t + \frac{10711}{241920}\right]$$

$$\beta_{\frac{1}{2}}(t) = \left[-\frac{162}{7}t^8 + \frac{648}{7}t^7 - \frac{726}{5}t^6 + \frac{558}{5}t^5 - \frac{127}{3}t^4 + \frac{20}{3}t^3 - \frac{17309}{181440}t + \frac{1567}{181440}\right]$$

$$\beta_{\frac{2}{3}}(t) = \left[\frac{243}{14}t^8 - \frac{459}{7}t^7 + \frac{963}{10}t^6 - \frac{2763}{40}t^5 + \frac{99}{4}t^4 - \frac{15}{4}t^3 + \frac{99}{4}t - \frac{163}{80640}\right]$$

$$\beta_{\frac{5}{6}}(t) = \left[-\frac{243}{35}t^8 + \frac{864}{35}t^7 - \frac{171}{5}t^6 + \frac{117}{5}t^5 - \frac{81}{10}t^4 + \frac{6}{5}t^3 - \frac{239}{17280}t + \frac{67}{120960}\right]$$

$$\beta_1(t) = \left[\frac{81}{70}t^8 - \frac{27}{7}t^7 + \frac{51}{10}t^6 - \frac{27}{8}t^5 + \frac{137}{120}t^4 - \frac{1}{6}t^3 + \frac{275}{145152}t - \frac{53}{725760}\right] \tag{7}$$

Estimating (7) at the non-interpolation points produces a discrete method of the form

$$\begin{bmatrix} y_{n+1} \\ y_{n+\frac{5}{6}} \\ y_{n+\frac{2}{3}} \\ y_{n+\frac{1}{6}} \\ y_n \end{bmatrix} - \begin{bmatrix} 4 & -3 \\ 3 & -2 \\ 2 & -1 \\ -1 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_{n+\frac{1}{3}} \end{bmatrix} = \frac{h^2}{2177280} \begin{bmatrix} 3966 & 62604 & 115938 & 163608 & 15138 & -108 & -66 \\ -159 & 4986 & 62379 & 101964 & 13239 & -1062 & 93 \\ 31 & -438 & 6513 & 48268 & 6513 & -438 & 31 \\ 31 & -186 & 213 & 5428 & 49353 & 5862 & -221 \\ -159 & 1206 & -4401 & 18804 & 96399 & 65718 & 3873 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+\frac{5}{6}} \\ f_{n+\frac{2}{3}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{1}{6}} \\ f_n \end{bmatrix} \tag{8}$$

The evaluation of the first derivative of (7) at all points give

$$\begin{bmatrix} y'_{n+1} \\ y'_{n+\frac{5}{6}} \\ y'_{n+\frac{2}{3}} \\ y'_{n+\frac{1}{2}} \\ y'_{n+\frac{1}{3}} \\ y'_{n+\frac{1}{6}} \\ y'_n \end{bmatrix} - \begin{bmatrix} 4354560 & -4354560 \\ 34836480 & -34836480 \\ 34836480 & -34836480 \\ 34836480 & -34836480 \\ 34836480 & -34836480 \\ 1451520 & -1451520 \\ 4354560 & -4354560 \end{bmatrix} \begin{bmatrix} y'_{n+\frac{1}{2}} \\ y'_{n+\frac{1}{3}} \end{bmatrix} = \frac{h}{580608} \begin{bmatrix} 294392 & 1412688 & 445320 & 1326176 & -154488 & 73296 & -10504 \\ -367 & 2073 & -828 & -89542 & -830271 & -51363 & 2618 \\ -367 & 2073 & -828 & -89542 & -830271 & -51363 & 2618 \\ -367 & 2073 & -828 & -89542 & -830271 & -51363 & 2618 \\ -367 & 2073 & -828 & -89542 & -830271 & -51363 & 2618 \\ 2808 & 20688 & -64680 & 46176 & -1084488 & -377904 & 11496 \\ 11000 & -80304 & 258696 & -553888 & -341112 & -1419696 & -293896 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+\frac{5}{6}} \\ f_{n+\frac{2}{3}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{1}{6}} \\ f_n \end{bmatrix} \tag{9}$$

Using matrix inversion on the methods in (8), yields the block method as seen in (10)

$$\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+\frac{5}{6}}\\y_{n+\frac{2}{3}}\\y_{n+\frac{1}{2}}\\y_{n+\frac{1}{3}}\\y_{n+\frac{1}{6}}\end{bmatrix}=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n-1}\\y_{n-\frac{5}{6}}\\y_{n-\frac{2}{3}}\\y_{n-\frac{1}{2}}\\y_{n-\frac{1}{3}}\\y_{n}\end{bmatrix}+h\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&\frac{5}{6}\\0&0&0&0&0&\frac{2}{3}\\0&0&0&0&0&\frac{1}{2}\\0&0&0&0&0&\frac{1}{3}\\0&0&0&0&0&\frac{1}{6}\end{bmatrix}\begin{bmatrix}y'_{n-1}\\y'_{n-\frac{5}{6}}\\y'_{n-\frac{2}{3}}\\y'_{n-\frac{1}{2}}\\y'_{n-\frac{1}{3}}\\y'_{n}\end{bmatrix}+$$

$$h^2\begin{bmatrix}0&0&0&0&0&\frac{41}{840}\\0&0&0&0&0&\frac{35225}{870912}\\0&0&0&0&0&\frac{272}{8505}\\0&0&0&0&0&\frac{253}{10752}\\0&0&0&0&0&\frac{1027}{68040}\\0&0&0&0&0&\frac{28549}{4354560}\end{bmatrix}\begin{bmatrix}f_{n-1}\\f_{n-\frac{5}{6}}\\f_{n-\frac{2}{3}}\\f_{n-\frac{1}{2}}\\f_{n-\frac{1}{3}}\\f_{n}\end{bmatrix}+h^2\begin{bmatrix}0&\frac{3}{70}&\frac{3}{280}&\frac{17}{105}&\frac{3}{140}&\frac{3}{13}\\-\frac{1375}{870912}&\frac{275}{20736}&-\frac{625}{96768}&\frac{25625}{217728}&\frac{3125}{29034}&\frac{8375}{48384}\\-\frac{2}{1701}&\frac{8}{945}&-\frac{2}{81}&\frac{656}{8505}&-\frac{2}{945}&\frac{376}{2835}\\-\frac{47}{53760}&\frac{57}{8960}&-\frac{363}{17920}&\frac{5}{128}&-\frac{267}{17920}&\frac{165}{1792}\\-\frac{19}{34020}&\frac{23}{5670}&-\frac{97}{7560}&\frac{197}{8505}&-\frac{2}{81}&\frac{97}{1890}\\-\frac{199}{870912}&\frac{403}{241920}&-\frac{7703}{1451520}&\frac{10621}{1088640}&-\frac{5717}{483840}&\frac{275}{20736}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+\frac{5}{6}}\\f_{n+\frac{2}{3}}\\f_{n+\frac{1}{2}}\\f_{n+\frac{1}{3}}\\f_{n+\frac{1}{6}}\end{bmatrix} \qquad (10)$$

Substituting the schemes that made up the block into equation (9), gives (11) below

$$\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y'_{n+1}\\y'_{n+\frac{5}{6}}\\y'_{n+\frac{2}{3}}\\y'_{n+\frac{1}{2}}\\y'_{n+\frac{1}{3}}\\y'_{n+\frac{1}{6}}\end{bmatrix}=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y'_{n-1}\\y'_{n-\frac{5}{6}}\\y'_{n-\frac{2}{3}}\\y'_{n-\frac{1}{2}}\\y'_{n-\frac{1}{3}}\\y'_{n}\end{bmatrix}+h\begin{bmatrix}0&0&0&0&0&\frac{41}{840}\\0&0&0&0&0&\frac{5491}{107520}\\0&0&0&0&0&\frac{5491}{107520}\\0&0&0&0&0&\frac{5491}{107520}\\0&0&0&0&0&\frac{5491}{107520}\\0&0&0&0&0&\frac{19087}{362880}\end{bmatrix}\begin{bmatrix}f_{n-1}\\f_{n-\frac{5}{6}}\\f_{n-\frac{2}{3}}\\f_{n-\frac{1}{2}}\\f_{n-\frac{1}{3}}\\f_{n}\end{bmatrix}$$

$$+h\begin{bmatrix}\frac{41}{840}&\frac{9}{35}&\frac{9}{280}&\frac{34}{105}&\frac{9}{280}&\frac{9}{35}\\-\frac{421}{215040}&\frac{1017}{71680}&-\frac{801}{17920}&\frac{8599}{107520}&-\frac{6039}{71680}&\frac{16893}{71680}\\-\frac{421}{215040}&\frac{1017}{71680}&-\frac{801}{17920}&\frac{8599}{107520}&-\frac{6039}{71680}&\frac{16893}{71680}\\-\frac{421}{215040}&\frac{1017}{71680}&-\frac{801}{17920}&\frac{8599}{107520}&-\frac{6039}{71680}&\frac{16893}{71680}\\-\frac{421}{215040}&\frac{1017}{71680}&-\frac{801}{17920}&\frac{8599}{107520}&-\frac{6039}{71680}&\frac{16893}{71680}\\-\frac{863}{362880}&\frac{263}{15120}&-\frac{6737}{120960}&\frac{293}{2835}&-\frac{15487}{120960}&\frac{2713}{15120}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+\frac{5}{6}}\\f_{n+\frac{2}{3}}\\f_{n+\frac{1}{2}}\\f_{n+\frac{1}{3}}\\f_{n+\frac{1}{6}}\end{bmatrix} \qquad (11)$$

Examination of the fundamental properties of the block method

We probe the order, local truncation error, and zero stability as well as the consistency of the block method which are the basic ingredient of a good method. This is demonstrated as follows:

Order and local truncation error

We describe a linear difference operators related to the block in (10), as

$$L\left[y(x):h\right]=\sum_{j=o}^{k}\left[\alpha_j y\left(x_n+jh\right)-h^2\beta_j f\left(x_n+jh\right)\right] \qquad (12)$$

where $y(x)$ is continuously differentiable function on $[a,b]$. Applying Taylor's series, expansion to (10), collecting their terms in powers of h produces

$$L[y(x):h] = C_0 y(x) + C_1 hy'(x) + C_2 hy''(x) + \cdots + O\left(h^{(q+1)}\right) \tag{13}$$

Where, C_i are constants. Then if the first C_{p+1} disappears, we have

$C_0 = C_1 = C_2 = \cdots, = C_p = C_{p+1} = 0$ and $C_{p+2} \neq 0$. Hence,

$$L[y(x):h] = C_{p+2} h^{p+2} y^{p+2}(x) + 0\left(h^{p+3}\right)$$

Where p is called the order of the method. Then $C_{p+2} h^{p+2} y^{p+2}(x)$ is the principal Local Truncation Error if the order p and error constant C_{p+2} are known (Lambert, 1973). From the foregoing, the block method in (10) is of a uniform order $[6 \quad 6 \quad 6 \quad 6 \quad 6]^T$ with Local Truncation Error of $\left(-\frac{1}{4761711360} \quad \frac{31}{609499054080} \quad -\frac{289}{219419659468800} \quad -\frac{31}{609499054080} \quad \frac{1}{4761711360}\right)^T$

Zero stability of the block

The block is said to be zero stable if the roots $z_{s,} s = 1, 2, ..., n$ of the characteristics polynomial $\rho(z)$ defined by $\rho(z) = \det(zA - E)$ satisfies $|z_s| \leq 1$ and the roots $|z_s| = 1$ is simple.

For the hybrid method,

$$A = \left[\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]$$

$$A = z^4 - z^3 = 0 \quad z = 0, 0, 0, 1$$

Therefore, the block is zero stable (Lambert, 1973).

Consistency

A block method is said to be consistent if it has order $p \geq 1$. Hence, our block method is consistent, since $p = 6$

Convergence of the block method

Theorem: The necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable (Lambert, 1973). By this theorem, the block method is convergent.

Numerical implementation and results

The proposed method (10) and (11) was used to solve three non-linear second order problems to assess the efficiency of the method. The results generated from the method is compared with some authors in literature.

Test Problem 1

$$y'' - x(y')^2 = 0 \qquad y(0) = 1, \ y'(0) = \frac{1}{2}, \ h = 0.05$$

Theoretical Solution: $y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$ (Ukpebor *et al.*, 2019; Adoghe and Omole, 2017).

Test Problem 2

$$y'' = \frac{(y')^2}{2y} - 2y \qquad y\left(\frac{\pi}{6}\right) = \frac{1}{4}, \ y'\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

Theoretical Solution: $y(x) = \sin^2 x$ (Ukpebor *et al.*, 2019; Odekunle *et al.*, 2014).

Test Problem 3

$$y'' + \left(\frac{6}{x}\right)y' + \left(\frac{4}{x^2}\right)y = 0 \qquad y(0) = 1, \ y'(0) = 1, \ h = \frac{0.1}{32}$$

Theoretical Solution: $y(x) = \frac{5}{3x} - \frac{2}{3x^4}$ (Ukpebor *et al.*, 2019; Badmus, 2014).

Numerical results and comparison

Table 1: Results and comparison of the result of the developed Method with the result of (Ukpebor *et al.*, 2019; Adoghe and Omole, 2017) for the Test Problem 1

x	Exact Result	Computational Result	Error	Error in Ukpebor *et al.*, (2019)	Error in Adoghe and Omole, (2017)
0.1	1.000260416672553500	1.000260416672553500	0.00E+00	1.11E-15	1.02E-11
0.2	1.001041667043427200	1.001041667043427000	2.22E-16	2.66E-15	8.00E-11
0.3	1.001302084069194000	1.001302084069193300	6.66E-16	5.10E-15	2.65E-10
0.4	1.001562501271567400	1.001562501271566100	1.33E-15	1.70E-14	6.24E-10
0.5	1.001822918685870000	1.001822918685868500	1.55E-15	2.66E-14	1.23E-09
0.6	1.002083336347423100	1.002083336347421300	1.77E-15	4.08E-14	2.18E-09
0.7	1.002343754291548500	1.002343754291546500	1.99E-15	6.03E-14	3.63E-09
0.8	1.002604172553569000	1.002604172553565500	3.55E-15	1.48E-13	5.81E-09
0.9	1.002864591168807200	1.002864591168799900	7.32E-15	1.73E-13	9.11E-09
1.0	1.003125010172585700	1.003125010172571500	1.42E-14	2.14E-13	1.41E-08

Table 2: Results and comparison of the result of the developed method with the result of (Ukpebor *et al.*, 2019; Jator, 2007) for the Test Problem 2

x	Exact Result	Computational Result	Error	Err in Error in Ukpebor *et al.*, (2019)	Error in Odekunle *et al.* (2014)
0.1	1.011979739721212900	1.011979739720540100	6.72E-13	3.45E-09	1.00E−09
0.2	1.012240194581662100	1.012240194580898500	7.63E-13	9.26E-09	1.03E−09
0.3	1.012500651102708600	1.012500651101789300	9.19E-13	1.67E-08	4.93E−09
0.4	1.012761109319706200	1.012761109318785600	9.20E-13	4.77E-08	9.03E−09
0.5	1.013021569268010000	1.013021569267086300	9.23E-13	5.11E-08	1.45E−08
0.6	1.013282030982976400	1.013282030982036700	9.39E-13	5.68E-08	1.99E−08
0.7	1.013542494499963400	1.013542494498971000	9.92E-13	6.41E-08	2.58E−08
0.8	1.013802959854330600	1.013802959853221400	1.10E-12	9.44E-08	3.11E−08
0.9	1.014063427081438400	1.014063427080129900	1.30E-12	9.77E-08	3.59E−08
1.0	1.014323896216649900	1.014323896215339600	1.30E-12	1.03E-07	4.30E−08

Table 3: Results and comparison of the result of the developed method with the result of (Ukpebor *et al.*, 2019; Badmus, 2014) for the Test Problem 3

x	Exact	Numerical	Error	Error in Ukpebor *et al.*, (2019)	Error in Badmus (2014)
0.1	1.490265439573534200	1.490265342533915800	9.70E-08	1.78E-06	8.30E−08
0.2	1.490593677537322700	1.490593579933485500	9.76E-08	2.41E-06	1.16E−06
0.3	1.492565167455561300	1.492565068284576800	9.91E-08	2.47E-06	6.63E−06
0.4	1.492894093872196400	1.492893994587302100	9.92E-08	2.57E-06	9.49E−06
0.5	1.493223119100596500	1.493223019697567300	9.94E-08	2.69E-06	1.95E−06
0.6	1.494210788815196500	1.494210688632888800	1.00E-07	3.18E-06	9.41E−06
0.7	1.494540210450245200	1.494540109688191400	1.00E-07	3.25E-06	4.65E−05
0.8	1.494869731477632200	1.494869630599486000	1.00E-07	3.34E-06	4.71E−05
0.9	1.495199352013941500	1.495199251015424400	1.00E-07	3.46E-06	1.86E−04
1.0	1.495529072175912700	1.495528971024403300	1.01E-07	3.94E-06	4.43E−04

Conclusion

This research has produced a new one-step implicit self-starting block method. The Numerical results produced from the tested problems are efficient and effective in performance. From the results generated, it is worthy of note the favorable performance of the proposed method when compared with existing methods in the literature. In Table 1, the results outperform that of (Ukpebor *et al.,* 2019; Adoghe and Omole, 2017) in terms of errors despite the higher order of the methods. Likewise, the better performance of the method was also showed in table 2 and table 3 when compared to (Ukpebor *et al.,* 2019; Badmus, 2014; Odekunle *et al.,* 2014). Therefore, this research has given rise to a new hybrid block method that is more accurate in solving non-linear second order initial value problems of ordinary differential equations directly.

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