ON THE FITTING ANALYSIS OF PERFORMANCE MEASURES FOR GENERAL PHASE TYPE DISTRIBUTIONS

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Abstract: The importance of phase type distribution in modeling activities cannot be under emphasized when both the first and second moments of a distribution are available or when the sequence of data points for computing moments is the information available. In continuous time process, phase – type distribution can be viewed to be the distribution of the time to absorption for an absorbing finite state Markov chain, and it is widely used in queueing theories and other fields of applied probabilities with the used of generalized Erlang, Coxian, Hypo-exponential, and Hyper-exponential distributions In this study, fitting analysis of general phase type distribution have been looked into, in order to provide meaningful study into the probability function, mean, k^{th} moment, variance, Laplace Stieltjes transform and squared coefficient of variation of phase type distribution. We begin from the tractability and memory less properties of exponential distribution, and since these properties are not enough, we looked into the passage through a phases of exponential stages by the use of matrix and vector operations to arrive at performance measures. Illustrative examples are demonstrated for various cases to arrive at various values for probability functions, Laplace Stieltjes transform, squared coefficient of variation,k^{th} moment, mean and variance for the phase type distribution.

Keywords: Coxian distribution, Erlang distribution, Hyper-exponential distribution, Hypo-exponential distribution, Phase type distribution**.**

Introduction

The exponential distribution is very important due to both its tractability and memoryless characteristics in performance modeling, but to overcome the model procedures these two properties may not be enough, and this makes the exponential distribution not sufficient. To model general distributions while sustaining the tractability property of the exponential, we make use of Phase type distribution. Also, phase - type distributions is very useful when the distribution with known mean and variance is to be formed, and the name phase-type distributions came to be due to fact that, processes can be represented as the movement through a succession of exponential stages. Phase type distribution has it major applications in queueing theories and applied probabilities with the used of generalized Erlang, Coxian, Hypo-exponential, and Hyper-exponential distributions. (Marie, 1980) obtained the useful technique in phase type distribution when subsystems are reduced to flow-equivalent servers representing the complementary network, while (Neuts, 1981) initiated the much cited article on phase-type distributions, and (Cumani, 1982; O'Cinneide, 1989) established some Important theoretical concepts on phase-type. (Ramaswami *et al.*, 1980; Ramaswami, 1988) derived a stable recursive scheme to compute the steady state probability vector.

(Aalen and Sidje, 1993) discussed the Hessenberg matrix computation of the exponential in the evaluation of the Padé approximation, while (Agboola, 2007) demonstrated the application of performance measures of M/G/1 queue system in solving the real life situation, and (Christian and Stephane, 2010) concentrated mainly on the finding of a Markov chain associated with some phase type distribution. The paper obtains some new results, and mentioned the problem left. Furthermore, (Agboola, 2010) applied the second order recurrence relation with constant coefficient, limiting behaviour and recursion process to arrive at performance measures, while (Aalen, 2014) provided the applications of phase type distribution in survival analysis. He analysed how a phenomenon such as a disease, moves through different phases, and calculated hazard rates and densities of phase-type distributions using Markov chain principle and affirmed that hazard rates are asymptotically constant due to quasi-stationarity.

(Agboola, 2016) determined the relative mix in the distribution of each machine type, and establishing a link between the durability and maintenance of each type of machine, using the repairman problem with multiple batch deterministic repairs of two different machine types, while (Belen *et al.*, 2020) developed a phase-type distribution approximation function so as to avoid the calculation of an inverse matrix. In addition, (Agboola, 2021) investigated the direct equation methods for the stationary distribution of Markov chains which produce a significantly more accurate response in less time for some types of situations when a specified number of well-defined stages have been completed. While (Agboola and Ayoade, 2021) obtained the probabilities of moving from transient states to one or more of the closed communicating classes and the probability of the absorption matrix, and (Agboola and Badmus, 2021) analysed the distribution function of the renewal process using the concept of discrete time Markov chain to obtain the performance measures which affirmed that, the Poisson process is the only renewal process with a linear mean value function. (Agboola and Obilade, 2022) considered the repairman problem with multiple batches of deterministic repairs of two different machine types, and established that, the closer the system is to a balanced situation the less is the discrepancy, while (Agboola and Ayinde, 2022). Developed an iterative solution methods for the stationary distribution of Markov chains and showed that, the block iterative method requires only a single iteration to obtain the solution, and (Osogami and Harchol-Balter, 2002) derived a partial characterization of the set of busy period durations which are presented by an r-phases Coxian distribution. (Bo Henry, 2022) extended the phase-type modeling to accommodate competing risks by using Coxian competing risks model and some data are evaluated. (Martin, 2023) Illustrated the multi-state processes models and when the dimension of the state space is greater than one obtained the proportional hazards specification.

(Yudong and Zhi-Scheng, 2023) used acyclic phase-type distributions (APHDs) in the canonical form to obtain a generic method for the two-layer censored data and formulated the expectation algorithm to compute the estimate by maximum likelihood. (Acal *et al*., 2024) built a non-homogeneous phase-type distribution for the hazard rate, reliability function, cumulative hazard rate using the maximum likelihood, and the characteristic function is evaluated. (*Hobolth et al.,2024)* demonstrated that phase-type distributions in coalescent models, by showing the 'phases' in the phase-type distribution correspond to states in the ancestral process, and concluded that, phase-type distributions enable a mathematical framework for coalescent theory. However, in this study, we established fitting analysis of general phase type distribution, in order to evaluate the mean, k^{th} *moment, variance, Laplace Stieltjes transform and squared coefficient of variation of phase type distributions.*

Notation

$E(Y)$, *expected value of random variable Y;* μ, *service time parameter;* σ_y^2, *variance;* $f_Y(y)$, *density function of random variable Y;* $F_Y(y)$, *distribution function of random variable Y;* $L_y(s)$, *Laplace transform of random variable Y;* p_k , probability of only the first k service phases being completed before the process is terminated; $E[Y^k]$, k^{th} moment of a random variable Y; $\propto_i$, the probability of moving from state i to state $(i+1)$; c_y^2, squared coefficient of variation of Random variable Y; R_i, $i = 1,2,3,\ldots,k$, initial probabilities; r_{ij}, $i,j = 1,2,3,\ldots,k$, routine probability.

Methodology

The study area emphasized the analysis of exponential distribution, two –exponential service phases, and processes, and general phase distribution, with the evaluation of fitting analysis of general phase type distribution to arrive at performance measure, Mean, variance, k^{th} moment, Laplace transform and squared coefficient of variation for general phase type distribution

The exponential -2 phase Distribution

Let Y denotes the random variable for service time of a customer at a service center, that exponential distributed with parameter $\mu > 0$ which consists of a single exponential phase is the time that the customer spends receiving service and this excluded the time the customer spends waiting for service. This is illustrated graphically in Figure 1 where the single exponential phase is denoted by a circle which contains the parameter of the exponential distribution. Customers are entering the stage from the left when amount of time which is exponentially distributed with parameter μ is spending within the stage and follow by customer exit to the right. If the random variable Y is chosen to represent the inter-arrival time of customers to a service center, then Y is exponentially distributed with mean $E(Y) = \frac{1}{\mu}$ and variance $\sigma_y^2 = \frac{1}{\mu^2}$

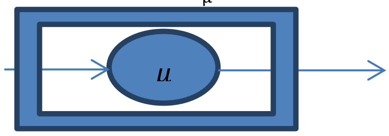

Figure 1: An Exponential Service Phase

The figure 1 indicates that, the service provided to a customer can be expressed as one exponential phase, while the figure 2 indicates that, the service time can be expressed by a second exponential phase.

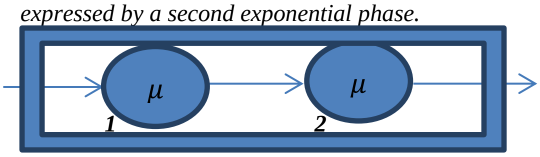

Figure 2: Two Exponential Service Phases in Tandem

With random variable Y the customer receives service which is exponentially distributed with parameter μ as the customer enters the servicing process. At the completion of service's stage, the customer enters the second stage when the exponentially distributed service time with parameter μ is begins with. At the second stage completion, another's customer service time is begins with by service process. Since both service stages are the same exponentially distributed with parameter μ, and they are independent. Then, two independent servers are not containing in the servicing process, but consist of a single service provider that operates in one stage or the other at a given time.

To analyze this situation, we shall assume that the probability density function of each of the phases is given by

$$f_Y(y) = \mu\, e^{-\mu y}, \qquad (1)$$

With

$$Expectation,\ E(Y) = \frac{1}{\mu}$$

And

$$Variance,\quad \sigma_y^2 = \frac{1}{\mu^2}.$$

The time that is randomly chosen from $f_Y(y)$ *is first spends by the customer. After the time completion, another amount of time chosen independently from* $f_Y(y)$ *is again spends. After the completion of this second time chosen, the new customer starts to receive service immediately the one in service departs. The total time distribution spent by the customer in the service is now looked into, and this is taking to be the random variable X which is the sum of two identically distributed and independent exponential random variables.*

Taking the exponentially random variable distributed with parameter μ *to be* Y.

Then

$$X = Y + Y. \qquad (2)$$

Therefore, using the convolution theorem relating to two random variables that are independent,

The convolution theorem states that

$$f_X(x) = \int_{-\infty}^{\infty} f_Y(y)\, f_Y(x-y)dy$$
$$= \int_0^x \mu\, e^{-\mu y}\, \mu\, e^{-\mu(x-y)}\, dy$$

$$\begin{cases} = \mu^2 e^{-\mu y}\int_0^x dy = \mu^2 x e^{-\mu y}, & x \geq 0 \\ 0, & x < 0 \end{cases} \quad (3)$$

The equation (3) represents the frequency density function for an Erlang-2, E_2 distribution, while the equation (4) below represents its cumulative distribution

$$F_X(x) = 1 - e^{-\mu x} - \mu x e^{-\mu x} = 1 - e^{-\mu x}\{1 + \mu x\}, \ x \geq 0. \quad (4)$$

The density function can equally be computed using Laplace transforms, by taking the Laplace transform of the total service time frequency density function as the product of the Laplace transform of the different stages.

Taking the Laplace transform of the total time in service distribution to be

$$L_X(s) = \int_0^{\infty} e^{-sx} f_X(x)dx$$

While the Laplace transform of each stage of the exponential phases be

$$L_y(s) = \int_0^{\infty} e^{-sy} f_Y(y)dy = \left(\frac{\mu}{s+\mu}\right)$$

Then

$$L_X(s) = E\left[e^{-s\{y_1+y_2\}}\right] = E[e^{-sy_1}] \times = E[e^{-sy_2}] = \left(\frac{\mu}{s+\mu}\right)^2 \quad (5)$$

To find the function of x whose transform is $\left(\frac{\mu}{s+\mu}\right)^2$, the Laplace inversion theorem is being used.

Since the Laplace transform of $\frac{1}{(s+a)^{r+1}}$, i..e. $L_X\left(\frac{1}{(s+a)^{r+1}}\right) = \frac{x^r}{r!}e^{-ax}$.

By representing a = μ and r = 1, we arrived at inversion of $L_X(s)$ to obtain

$$f_X(x) = \mu^2 x e^{-\mu y}, \quad x \geq 0$$

Likewise, we may obtain the mean value and higher moments from the Laplace transform as

$$E[X^k] = (-1)^k \frac{d^k}{ds^k} L_X(s)\bigg|_{s=0}, \quad for\ k = 1, 2, \cdots$$

$$E[X] = \frac{d}{ds} L_X(s)\Big|_{s=0} = -\mu^2 \frac{d}{ds}\left(\frac{1}{s+\mu}\right)^2 = \mu^2 \frac{d}{ds}(s+\mu)^{-2}\Big|_{s=0} = \frac{2}{\mu}. \quad (6)$$

$$\sigma_X^2 = \left(\frac{1}{\mu}\right)^2 + \left(\frac{1}{\mu}\right)^2 = \frac{2}{\mu^2}. \quad (7)$$

General Phase-Type Distributions

Thee general phase type distribution consist of a collection of k phases such that the random variable that describes the time spent in phase $i, i = 1,2,\ldots,k$, is exponentially distributed with parameter μ_i. It is possible to envisage a general phase type probability distribution defined on these phases as the total time spent moving in some probabilistic fashion among the k different phases.

To analyse general phase type distribution, we make the following representation:

- For the initial probabilities, let, R_i, $i = 1, 2, 3, \ldots, k$, denote the probabilities that a given phase i is the first phase entered: $\sum_{i=1}^{k} R_i = 1$.
- For the routine probabilities, let r_{ij}, $i, j = 1, 2, 3, \ldots, k$, denotes the probability that, after spending an exponentially distributed amount of time with mean $\frac{1}{\mu_i}$ in phase i, the next phase entered is phase j, for all i. $r_{ii} < 1$, while for at least a value of i, $\sum_{i=1}^{k} r_{ij} < 1$.
- For the terminal probabilities, let φ_i, $i = 1, 2, 3, \ldots, k$, denote the probabilities that the process terminates on exiting from phase i, and at least one must be strictly positive. For all i, $i = 1, 2, 3, \ldots, k$,

Therefore,

$$\varphi_i + \sum_{i=1}^{k} r_{ij} = 1:$$

This implies that, on exiting from phase i, either another phase j is entered with probability r_{ij} or the process terminates with probability φ_i

Therefore, we can analyse the Coxian distribution as form of general type distribution as follow:

Given the Coxian distribution with the initial probabilities vector $R = (1,0,0,\ldots,0)$; and the terminal probabilities $\varphi = (1-\alpha_1), (1-\alpha_2), (1-\alpha_3), \ldots, (1-\alpha_{k-1}), 1))$

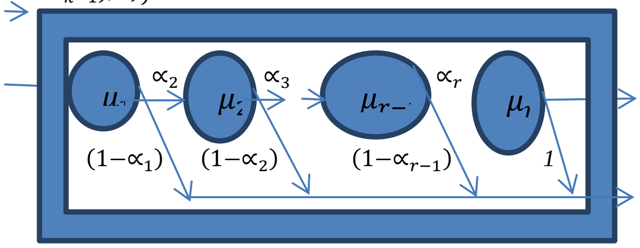

Figure 3: Coxian Distribution

The probabilities r_{ij} are the elements of the matrix

$$M = \begin{pmatrix} 0 & \propto_1 & 0 & \cdots & 0 \\ 0 & 0 & \propto_2 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \propto_{k-1} \\ 0 & 0 & 0 & & 0 \end{pmatrix}$$

The vectors $R$and φ and the matrix M together with the parameters of the exponential distributions completely characterize a Coxian distribution.

The following two cases are going to be considered:

i. When the squared coefficient of variation, $c_y^2 \leq 1$;
ii. When the squared coefficient of variation, $c_y^2 > 1$

To solve the first case, $c_y^2 \leq 1$, we assumed the following

$$\mu_i = \mu, \quad \forall\, i = 1, 2, \ldots, r$$

And

$$\alpha_i = \alpha, \quad \forall\, i = 1, 2, \ldots, r$$

This is shown in the figure 4 below:

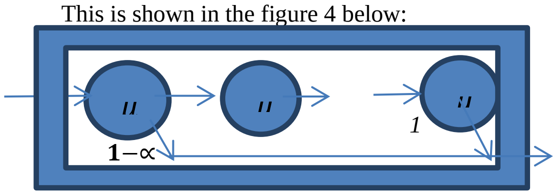

Figure 4: Coxian for $c_y^2 < 1$

To obtain the value that we must assign to μ and ∝, we must first compute the mean and the square coefficient of variation of the distribution represented by figure 4.

We begin with Laplace transform

$$L_Y(s) = (1-\propto)\frac{\mu}{s+\mu} + \propto \prod_{i=1}^{r} \frac{\mu}{s+\mu} = (1-\propto)\frac{\mu}{s+\mu} + \propto \frac{\mu^r}{s+\mu^r}$$

$$E[Y] = \frac{d}{ds}\left((1-\alpha)\frac{\mu}{s+\mu} + \alpha\frac{\mu^r}{s+\mu^r}\right)_{s=0}$$
$$E[Y] = \left[(1-\alpha)\frac{\mu}{(s+\mu)^2} + \alpha\frac{\mu^r r}{(s+\mu)^{r+1}}\right]_{s=0}$$
$$E[Y] = (1-\alpha)\frac{1}{\mu} + \alpha\frac{r}{\mu} \qquad (14)$$

$$E[Y^2] = (-1)^2\frac{d^2}{ds^2}\left((1-\alpha)\frac{\mu}{s+\mu} + \alpha\frac{\mu^r}{s+\mu^r}\right)_{s=0}$$
$$= \frac{d}{ds}\left(-(1-\alpha)\frac{\mu}{(s+\mu)^2} - \alpha\frac{\mu^r r}{(s+\mu)^{r+1}}\right)_{s=0}$$
$$\left((1-\alpha)\frac{2\mu}{(s+\mu)^3} - \alpha\frac{r(r+1)}{(s+\mu)^{r+2}}\right)_{s=0}$$
$$E[Y^2] = (1-\alpha)\frac{2}{\mu^2} + \alpha\frac{r(r+1)}{\mu^2} \qquad (15)$$

$$Var[Y] = E[Y^2] - [E[Y]]^2 = (1-\alpha)\frac{2}{\mu^2} + \alpha\frac{r(r+1)}{\mu^2} - \left[(1-\alpha)\frac{1}{\mu} + \alpha\frac{r}{\mu}\right]^2$$
$$Var[Y] = \frac{2(1-\alpha) + \alpha r(r+1) - (1-\alpha+ar)^2}{\mu^2}$$

Finally, we obtain the coefficient as $c_y^2 = \frac{Var[Y]}{E[Y]^2} = \frac{2(1-\alpha)+\alpha r(r+1)-(1-\alpha+ar)^2}{\mu^2} \Big/ (1-\alpha)\frac{2}{\mu^2} + \alpha\frac{r(r+1)}{\mu^2}$

$$c_y^2 = \frac{2(1-\alpha)+\alpha r(r+1)-(1-\alpha+ar)^2}{(1-\alpha+ar)^2} \qquad (16)$$

The next step is to select three parameters, r, α, and μ, that satisfy the two equations (14) and (16). Additionally, we are to choose r to be greater than $\frac{1}{c_y^2}$, since it is advantageous to choose r as small as possible. Therefore, setting

$$r = \frac{1}{c_y^2}$$

Having chosen a value for r, and with our given value of c_y^2, we can use Equation (16), which involves only r, c_y^2, and α, to find an appropriate value for α. Marie (1980) recommends the choice that,

$$\alpha = \frac{r - 2c_y^2 + \sqrt{r^2 + 4 - 4rc_y^2}}{2(c_y^2+1)(r-1)}$$

Finally, μ is then computed from Equation (14).

For the second case, $c_y^2 > 1$.

Let us begins from Coxian for $c_y^2 \geq 0.5$ as denoted in the Figure 5 below:

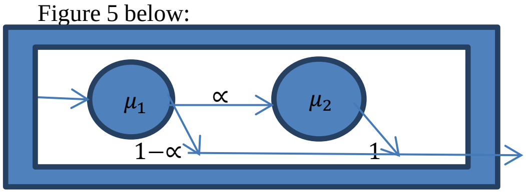

Figure 5: Coxian for $c_y^2 \geq 0.5$

Where for Cox-2 distribution

$$E[Y] = \frac{1}{\mu_1} + \frac{\alpha}{\mu_2} = \frac{\mu_2 + \alpha\mu_1}{\mu_1\mu_2}.$$

And

$$Var[Y] = \frac{\mu_2^2 + \alpha\,{\mu_1}^2(2-\alpha)}{(\mu_2 + \alpha\mu_1)^2}$$

Given $E[Y]$and $\boldsymbol{c_y^2}$, our task is to use these equations to find the three parameters μ_1, μ_2, and α. From the infinite number of solutions possible, the following is frequently recommended since it yields particularly simple forms by Marie (1980):

$$\mu_1 = \frac{2}{E[Y]}, \quad \alpha = \frac{1}{2\boldsymbol{c_y^2}}, \quad and \;\; \mu_2 = \frac{1}{E[Y]\boldsymbol{c_y^2}} = \alpha\,\mu_1.$$

This distribution is valid for values of $\boldsymbol{c_y^2}$ that satisfy $\boldsymbol{c_y^2 \geq 0.5}$, and not just for those values greater than or equal to 1.

Results and Discussion

This section presented the solutions for fittings of performance measuresfor general phase type distributions to obtain the expectation, kth moment, variance, and squared coefficient of variation, as well as its probability density function.

Illustrative Example 1

Given a general phase-type distribution having four phases and exponential parameters $\mu_i, i = 1,2,3,4$; vectors$R = (0,.4,0,.6)$ and $\varphi = (0,0,.1,0)$ and routing probability matrix

$$M = \begin{pmatrix} 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \\ 0.2 & 0 & 0 & 0.7 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

begins life in phase 2 with probability 0.4 or in phase 4 with probability 0.6, and moves among the phases until it eventually departs from phase 3. It has the graphical interpretation of Figure 6.

General phase-type distributions frequently have an extra phase appended to represent the exterior into which the process finally departs. Once in this phase, called a sink or an absorbing phase, the process remains there forever. In this case, it is usual to combine the parameters of the exponential distributions of the phases and the routing probabilities into a single matrix Q whose elements q_{ij} give the rate of transition

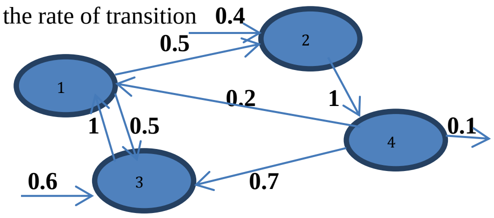

Figure 6: A general Phase Type Distribution

from phase i to some other phase j, i.e., $q_{ij} = \mu_{ij}r_{ij}$. It is also usual to set the diagonal element in each row of this matrix equal to the negated sum of the off-diagonal elements in that row, i.e., $q_{ii} = \sum_{i\neq j} q_{ij}$ Thus the sum of all the elements in any row is zero. As we shall see later, this matrix together with an initial starting vector describes a continuous-time Markov chain with a single absorbing state. The matrix is called the transition-rate matrix or infinitesimal generator of the Markov chain. For given Initial distribution: $(1 \quad 0 \quad 0 \quad \ldots \quad 0 \quad 0) = (R \quad O)$

$$M = \begin{pmatrix} -\mu_1 & \mu_1 \propto_1 & 0 & \cdots & \mu_1(1-\propto_1) \\ 0 & \mu_2 & \mu_2 \propto_2 & & \mu_2(1-\propto_2) \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \mu_{k-1}\propto_{k-1} & \mu_{k-1}(1-\propto_{k-1}) \\ & & & \cdots \; -\mu_k & \mu_k \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now, given a Coxian distribution with four phases and rates $\mu_1 = 1, \mu_2 = 2, \mu_3 = 4, \mu_4 = 8$. On completion of phase $i = 1,2,3$, the process proceeds to phase $i+1$ with probability 0.5 or enters the sink phase with probability 0.5. This allows us to write.

Solution

$$\boldsymbol{R^I} = (\mathbf{1} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0} \quad \mathbf{0}) = (\boldsymbol{R} \quad \mathbf{0})$$

$$\boldsymbol{M} = \begin{pmatrix} -\mu_1 & 0.5\mu_1 & \mathbf{0} & \mathbf{0} & 0.5\mu_1 \\ \mathbf{0} & -\mu_2 & \mathbf{0.5}\mu_2 & \mathbf{0} & \mathbf{0.5}\mu_2. \\ \mathbf{0} & 0 & -\mu_3 & \mathbf{0.5}\mu_3 & \mathbf{0.5}\mu_3 \\ 0 & \mathbf{0} & \mathbf{0} & -\mu_4 & \mu_4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} -1 & 0.5 & \mathbf{0} & \mathbf{0} & 0.5 \\ \mathbf{0} & -2 & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & -4 & \mathbf{2} & \mathbf{2} \\ 0 & \mathbf{0} & \mathbf{0} & -8 & 8 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{S} & \boldsymbol{S^0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

where

$$S^{-1} = \begin{pmatrix} -1 & -0.25 & -0.0625 & -0.015625 \\ 0 & -0.5 & -0.125 & -0.03125 \\ 0 & 0 & -0.25 & -0.0625 \\ 0 & 0 & 0 & -0.125 \end{pmatrix}$$

and

$$-RS^{-1} = 1 \quad 0.25 \quad 0.0625 \quad 0.1563.$$

Therefore the expectation of this Coxian (and the mean time to absorption in the Markov chain) is

$$-RS^{-1}e = 1.3281$$

Also, the mean time spent in

Phase 1: $\frac{1}{\mu_1} = 1 = y_1$

Phase 2: $\frac{0.5}{\mu_2} = 0.25 = y_2$

Phase 3: $\frac{0.5 \times 0.5}{\mu_3} = 0.0625 = y_3$

Phase 4: $\frac{(0.5)^3}{\mu_4} = 0.015625 = y_4$

Fitting Phase-Type Distributions to Means and Variances

Illustrative Example 2

To construct a phase-type distribution having expectation $E[Y] = 3$ and variance $Var[Y] = 4$. With these parameters, we have $c_y^2 = \frac{4}{9} = 0.4444$ which is less than 1 and we may use the analysis just developed. We choose parameters for a Coxian distribution as represented in Figure 7:

$$r = \left\lceil \frac{1}{c_y^2} \right\rceil = \frac{1}{0.4444} = 2.25$$

$$\propto = \frac{r - 2c_y^2 + \sqrt{r^2 - 4 - 4rc_y^2}}{2(c_y^2+1)(r-1)}$$

$$= \frac{2.25 - 2(0.4444) + \sqrt{(5.0626 + 4 - 4(2.25)(0.4444))}}{2(0.4444+1)(2.25-1)}$$

$$\propto = \frac{1.3612 + \sqrt{1.063}}{3.611} = 0.6625$$

$$\mu = \frac{1 + \propto (r-1)}{E[Y]} = \frac{1 + 0.6625(2.25-1)}{3} = \frac{1.82813}{3} = 0.6094$$

Let us check our answers by computing the expectation and variance of this Coxian:

$$E[Y] = (1-\propto)\frac{1}{\mu} + \propto \frac{r}{\mu}$$

$$= (1 - 0.6625)\frac{1}{0.6094} + 0.6625\left(\frac{2.25}{0.6094}\right)$$

$$= 0.5538 + 2.446 = 3$$

$$Var[Y] = \frac{2(1-\propto) + \propto r(r+1) - (1-\propto + ar)^2}{\mu^2}$$

$$= \frac{2(1-0.6625) + 0.6625(2.25)(3.25) - (1 - 0.6625 + 1.4906)^2}{0.3714}$$

$$Var[Y] = \frac{0.675 + 4.8445 - 3.3420}{0.3714} = \frac{2.1776}{0.3714} = 5.8632.$$

For coefficients of variation greater than 1, it suffices to use a two-phase Coxian. This is represented in Figure 7, where it is apparent that we need to find three parameters, namely, μ_1, μ_2 *and* α.

Illustrative Example 3

A random variable Y having expectation $E[Y] = 3$and standard deviation equal to $\sigma_Y = 4$ may be modeled as a two-phase Coxian. Given that $E[Y] = 3$and $\boldsymbol{c_y^2} = \frac{\mathbf{16}}{\mathbf{9}}$, we may take the parameters of the Coxian distribution to be

$$\mu_1 = \frac{2}{E[Y]} = \frac{2}{3}, \quad \propto = \frac{1}{2\boldsymbol{c_y^2}} = \frac{9}{32}, \quad and \ \mu_2 = \frac{1}{E[Y]\boldsymbol{c_y^2}} = \propto \mu_1 = \frac{3}{16}.$$

To check, we find the expectation and standard deviation of this Coxian

$$E[Y] = \frac{1}{\mu_1} + \frac{\propto}{\mu_2} = \frac{\mu_2 + \propto\mu_1}{\mu_1\mu_2} = \frac{3/16 + 9/32(2/3)}{2/3(3/16)} = \frac{6/16}{2/16} = 3.$$

And

$$Var[Y] = \frac{\mu_2^2 + \propto {\mu_1}^2(2-\propto)}{(\mu_2 + \propto \mu_1)^2} = \frac{\left(\frac{3}{16}\right)^2 + \left(\frac{2}{3}\right)^2\left(2 - \frac{9}{32}\right)}{\left(\frac{3}{16} + \frac{3}{16}\right)^2} = \frac{16}{9}$$

As an alternative to the Coxian-2, a two-phase hyper-exponential distribution such as that shown in Figure13 may also be used to model distributions for which squared coefficient of variation,$\boldsymbol{c_y^2} \geq \mathbf{1}$. As is the case for a Coxian-2 distribution, a two-phase hyper-exponential distribution is not defined uniquely by its first two moments so that there is a considerable choice of variations possible. One that has attracted attention is to balance the means by imposing the additional condition

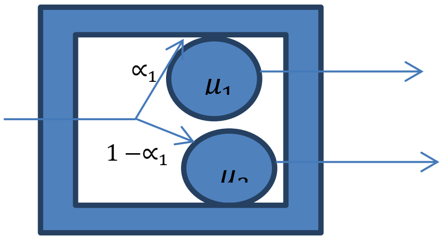

Figure 7: Two Phase Hyper-exponential Distribution

$$\frac{\propto}{\mu_1} = \frac{1-\propto}{\mu_2}$$

This leads to the formulae

$$\propto = \frac{1}{2}\left(1 + \sqrt{\frac{\boldsymbol{c_y^2 - 1}}{\boldsymbol{c_y^2 + 1}}}\right), \quad \mu_1 = \frac{2\propto}{E[Y]}, \quad and\ \mu_2 = \frac{2(1-\propto)}{E[Y]}.$$

Illustrative Example 3

Let us return to the previous example of a random variable with expectation $E[Y] = 3$ and squared coefficient of variation $\boldsymbol{c_y^2 = \frac{16}{9}}$ and see what parameters we obtain for the two-phase hyper-exponential:

$$\propto = \frac{1}{2}\left(1 + \sqrt{\frac{\frac{16}{9} - 1}{\frac{16}{9} + 1}}\right) = 0.7646,$$

$$\mu_1 = \frac{2\propto}{E[Y]} = \frac{2(0.7646)}{3} = 0.5097.$$

$$\mu_2 = \frac{2(1-\propto)}{E[Y]} = \frac{2(1 - 0.7646)}{3} = 0.1570.$$

We now check these results. The expectation of the two-phase hyper-exponential is given by

$$E[Y] = \frac{\propto}{\mu_1} + \frac{1-\propto}{\mu_2} = \frac{0.7646}{0.5097} + \frac{0.2354}{0.1570} = 3.0$$

Similarly, the squared coefficient of variation is given by

$$c_y^2 = \frac{2\propto/{\mu_1}^2 + 2(1-\propto)/{\mu_2}^2}{\left(\propto/\mu_1 + (1-\propto)/\mu_2\right)^2} - 1$$

$$= \frac{2(0.7646)/(0.5097)^2 + 2(0.2354)/(0.1570)^2}{\left(0.7646/0.5097 + (0.2354)/0.1570\right)^2} - 1$$

$$= \frac{25}{9} - 1 = \frac{16}{9}$$

In this study, we considered the possibility of matching only the first two moments of a distribution with phase-type representations, but these could be extendedto matching higher moments.

Conclusion:

In this study, The fittings of Performance measures for general phase type distribution is considered, in order to provide some insight into the mean, k^{th} moment, variance, Laplace Stieltjes transform and squared coefficient of variation of phase type distribution.we begin from the exceptional mathematical tractability and memoryless property of exponential distribution, and since these properties of exponential are not enough, we considered the passage through a succession of exponential phases or stages through matrix and vector operations to arrive at performance measures. Illustrative examples are demonstrated for various cases to arrive at various values for Laplace Stieltjes transform, squared coefficient of variation,k^{th} moment, mean and variance for the phase type distributions.

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